1	2	3	4	$\sum$

BROJ INDEKSA

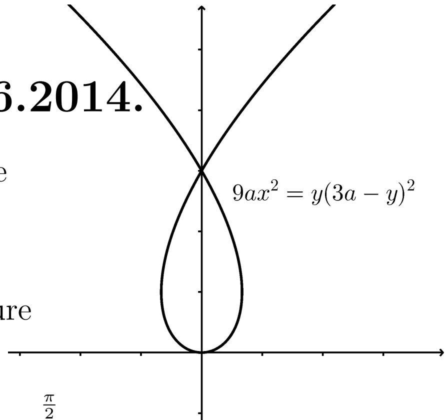
SMJER STUDIJA

IME I PREZIME

Grupa A

OBRATITI PAŽNJU NA MATEMATIČKU KULTURU I MATEMATIČKU PISMENOST

Matematika II, pismeni ispit, 16.06.2014.



1. Izračunati površinu površi koja nastaje kada petlja krive $9ax^2 = y(3a-y)^2$ rotira oko y-ose (misli se na dio krive za koju je $0 \leq y \leq 3a$).

2. Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^2 = 2y^3$.

3. Metodom diferenciranja po parametru izračunati integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\operatorname{arc\,tg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\mathrm{d}x$ $(\alpha > 0)$ pa

dobijeni rezultat iskoristiti i izračunati $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x}\mathrm{d}x$.

4. Izračunati integral $I = \displaystyle\iint_S z^3\mathrm{d}x\mathrm{d}y + x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}z\mathrm{d}x$ gdje je S-vanjska strana konusne površi $G: x^2+y^2 \leq z^2$, $0 \leq z \leq 1$.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

1	2	3	4	$\sum$

BROJ INDEKSA

SMJER STUDIJA

IME I PREZIME

Grupa B

OBRATITI PAŽNJU NA MATEMATIČKU KULTURU I MATEMATIČKU PISMENOST

Matematika II, prvi parcijalni, 28.04.2014.

1. Izračunati površinu omotača tijela koje nastaje kada luk kruga $x^2+(y-b)^2 = R^2$ rotira oko y-ose, i to luk kruga koji se nalazi između pravih $y = y_1$ i $y = y_2$ (gdje su $y_1 > b > y_2 > 0$).

2. Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^3 = a^2(x^4+y^4)$.

3. Metodom diferenciranja po parametru izračunati integral $\displaystyle\int_0^{\infty} \frac{\operatorname{arc\,tg}\alpha x}{x(1+x^2)}\mathrm{d}x$, gdje je $\alpha > 0$.

4. Izračunati integral $I = \displaystyle\iint_W y^3\mathrm{d}z\mathrm{d}x + z^3\mathrm{d}x\mathrm{d}y + x^3\mathrm{d}y\mathrm{d}z$ gdje je S-vanjska strana konusne površi $H: x^2+y^2 \leq z^2$, $0 \leq z \leq 1$.

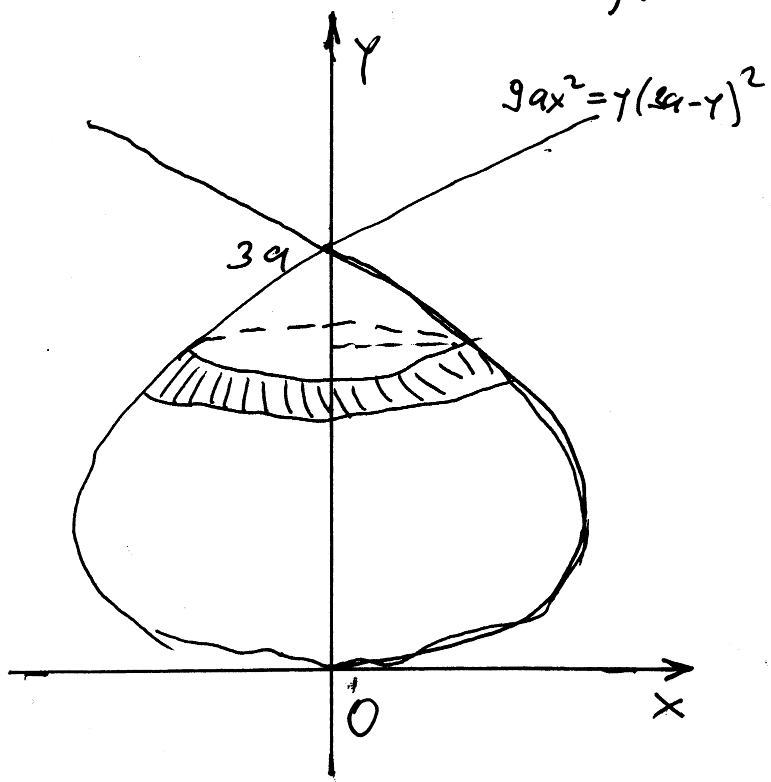
VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

Zadaci su skinuti sa stranice ff.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

Izračunati površinu površi koja nastaje kada petlja krive $9ax^2 = y(3a-y)^2$ rotira oko y-ose (misli se na dio krive za koju je $0 \leq y \leq 3a$).

Rj.

Skica petlje krive je



Koristićemo sljedeću formulu

$x = x(y), \quad y_1 \leq y \leq y_2$

$$P = 2\pi \int_{y_1}^{y_2} x\sqrt{1+x'^2}\, dy$$

$$9ax^2 = y(3a-y)^2 \quad /'$$

$$18a\,xx' = (3a-y)^2 - 2y(3a-y) = 3(3a-y)(a-y) \Rightarrow xx' = \frac{(3a-y)(a-y)}{6a}$$

Sad imamo

$$P = 2\pi \int_0^{3a} \sqrt{x^2 + (xx')^2}\, dy = 2\pi \int_0^{3a} \sqrt{\frac{y}{9a}(3a-y)^2 + \frac{1}{36a^2}(3a-y)^2(a-y)^2}\, dy$$

$$= 2\pi \int_0^{3a} \frac{1}{6a}(3a-y)\sqrt{a^2 + 2ay + y^2}\, dy = \frac{\pi}{3a}\int_0^{3a}(3a^2 + 2ay - y^2)\, dy = 3\pi a^2.$$

(#) Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^2=2y^3$.

Rj: Izgled date linije ne igra nikakvu ulogu u rješavanju ovog zadatka.

$$P=\iint_D dx\,dy$$

uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$x^2+y^2=\rho^2$

$$(x^2+y^2)^2=2y^3$$

$$(\rho^2)^2=2\rho^3\sin^3\varphi$$

$$\rho^4=2\rho^3\sin^3\varphi \quad /:\rho^3 \ (\rho\neq 0)$$

$$\rho=2\sin^3\varphi$$

jednakost je definisana za $\forall\varphi\in(0,\pi)$

ova jednakost nema smisla kada je $\varphi\in(\pi,2\pi)$

$$D \xrightarrow{\text{transformise}} D':\begin{cases} 0\le\varphi\le\pi \\ 0\le\rho\le 2\sin^3\varphi \end{cases}$$

$$P=\iint_{D'}\rho\,d\rho\,d\varphi=\int_0^{\pi}d\varphi\int_0^{2\sin^3\varphi}\rho\,d\rho=\int_0^{\pi}\frac{1}{2}\rho^2\Big|_0^{2\sin^3\varphi}d\varphi=$$

$$=\frac{1}{2}\int_0^{\pi}4\sin^6\varphi\,d\varphi=2\int_0^{\pi}\sin^6\varphi\,d\varphi=\ldots=2\cdot\frac{5\pi}{16}=\frac{5\pi}{8}$$

tražena površina

(#) Metodom diferenciranja po parametru izračunati integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\operatorname{arctg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\,dx$ pa dobijeni rezultat iskoristiti i izračunati $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x}\,dx$ $(\alpha > 0)$.

Rj.

$$J(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\operatorname{arctg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\,dx \qquad \left(\frac{\operatorname{arctg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\right)'_{\alpha} = \frac{1}{\operatorname{tg} x} \cdot \frac{\operatorname{tg} x}{1+\alpha^2 \operatorname{tg}^2 x}$$

$$J' = \int_0^{\pi/2} \left(\frac{\operatorname{arctg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\right)'_{\alpha} dx \qquad = \frac{1}{1+\alpha^2 \operatorname{tg}^2 x}$$

$$J' = \int_0^{\pi/2} \frac{dx}{1+\alpha^2 \operatorname{tg}^2 x} = \left| \begin{array}{ll} \operatorname{tg} x = z & \\ x = \operatorname{arctg} z & x\Big|_0^{\frac{\pi}{2}} \Rightarrow z\Big|_0^{\infty} \\ dx = \dfrac{dz}{1+z^2} & \end{array} \right| =$$

$$= \int_0^{\infty} \frac{dz}{(1+\alpha^2 z^2)(1+z^2)} \overset{(*)}{=\!=}$$

$$\frac{1}{(1+\alpha^2 z^2)(1+z^2)} = \frac{Az+B}{1+\alpha^2 z^2} + \frac{Cz+D}{1+z^2} \quad \Big/ (1+\alpha^2 z^2)(1+z^2)$$

$$1 = Az(1+z^2) + B(1+z^2) + Cz(1+\alpha^2 z^2) + D(1+\alpha^2 z^2)$$

z^3: $A + \alpha^2 C = 0$

z^2: $B + \alpha^2 D = 0$

z: $A + C = 0$

z^0: $B + D = 1$

$$\Rightarrow \dots \Rightarrow A = 0,\ B = \frac{\alpha^2}{\alpha^2 - 1},\ C = 0,\ D = \frac{-1}{\alpha^2 - 1}$$

$$\overset{(*)}{=} \frac{\alpha^2}{\alpha^2-1}\int_0^\infty \frac{dz}{1+\alpha^2 z^2} + \frac{(-1)}{\alpha^2-1}\int_0^\infty \frac{dz}{1+z^2} =$$

$$= \frac{\alpha^2}{\alpha^2-1}\cdot\frac{1}{\alpha^2}\int_0^\infty \frac{dz}{\left(\frac{1}{\alpha}\right)^2+z^2} + \frac{(-1)}{\alpha^2-1}\int_0^\infty \frac{dz}{1+z^2}$$

$$= \frac{1}{\alpha^2-1}\cdot\alpha \operatorname{arc\,tg} \alpha z\Big|_0^\infty + \frac{(-1)}{\alpha^2-1}\operatorname{arc\,tg} z\Big|_0^\infty$$

$$= \frac{\alpha}{\alpha^2-1}\left(\frac{\pi}{2}-0\right) + \frac{(-1)}{\alpha^2-1}\left(\frac{\pi}{2}-0\right) = \frac{\frac{\pi}{2}}{\alpha+1}$$

Dobili smo $J'_\alpha = \frac{\pi}{2}\cdot\frac{1}{\alpha+1} \Rightarrow J = \frac{\pi}{2}\ln|\alpha+1| + C$

Kako je $J(0) = 0 = \frac{\pi}{2}\ln 1 + C \Rightarrow C = 0$

pa imamo

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{arc\,tg}(\alpha \operatorname{tg} x)}{\operatorname{tg} x}\,dx = \frac{\pi}{2}\ln|\alpha+1|$$

Za $\alpha = 1$

$$\int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x} = \frac{\pi}{2}\ln 2 .$$

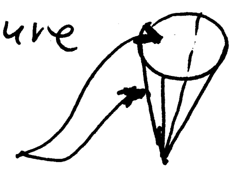
Izračunati integral

$$I=\iint\limits_S x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy$$

gdje je S - vanjska strana konusne površi $G: x^2+y^2\le z^2$, $0\le z\le 1$.

Rj. - upute:

Označimo sa I_1 integral po površini S_1 cijele date figure



S_1

(omotača i baze) a sa I_2 integral po gornjoj strani baze S_2. Tada je $I=I_1-I_2$. U integralu I_1 primjenom formule Gauss-Ostrogradskog

$$I_1=3\iiint\limits_\Omega (x^2+y^2+z^2)\,dx\,dy\,dz$$

Prelazeći na cilindrične koordinate dobićemo sljedeće

$$I_1=3\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^z(\rho^2+z^2)\rho\,d\rho=\ldots=\frac{9}{10}\pi$$

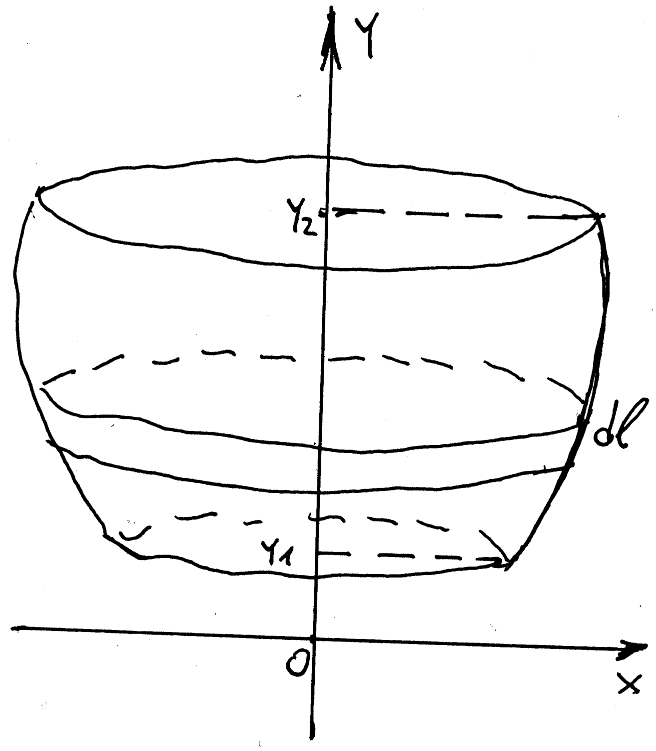
Računajući integral po osnovici konusa

$$I_2=\iint\limits_{S_2} x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy=\iint\limits_{S_2}dx\,dy=\pi$$

Prema tome $I=-\frac{\pi}{10}$ tražena vrijednost.

(#) Izračunati površinu omotača tijela koje nastaje kada luk kruga $x^2+(y-b)^2=R^2$ rotira oko y-ose, i to luk kruga koji se nalazi između pravih $y=y_1$ i $y=y_2$ (gdje su $y_1, y_2, b>0$).

Rj.



U ovom slučaju precizna slika ne igra veliku ulogu u rješavanju zadatka pa nacrtajmo sliku bez preciznih dimenzija.

Prisjetimo se formule

$$P=2\pi\int_{t_1}^{t_2}|\mu(t)|\sqrt{\dot{\eta}^2+\dot{\mu}^2}\,dt$$

za $C:\begin{cases} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{cases}$

računanje površine omotača kada tijelo rotira oko x-ose

Za $x=x(y)$, $y_1\le y\le y_2$

$$P=2\pi\int_{y_1}^{y_2} x\sqrt{1+x'^2}\,dy$$

$$x^2+(y-b)^2=R^2 \quad /'$$

$$2xx'+2(y-b)=0$$

$$xx'=-(y-b)$$

$$P=2\pi\int_{y_1}^{y_2} x\sqrt{1+(x')^2}\,dy=2\pi\int_{y_1}^{y_2}\sqrt{x^2+(xx')^2}\,dy=2\pi\int_{y_1}^{y_2}\sqrt{R^2-(y-b)^2+(y-b)^2}\,dy$$

$$=2\pi R\int_{y_1}^{y_2}dy=2\pi R(y_2-y_1)$$ traženo rješenje

(#) Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^3 = a^2(x^4+y^4)$.

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$x=\rho\cos\varphi$

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$x^2+y^2=\rho^2$

$$(x^2+y^2)^3=a^2(x^4+y^4)$$

$$(\rho^2)^3=a^2(\rho^4\cos^4\varphi+\rho^4\sin^4\varphi)$$

$$\rho^6=a^2\rho^4(\cos^4\varphi+\sin^4\varphi) \quad /:\rho^4$$

$$\rho^2=a^2(\cos^4\varphi+\sin^4\varphi)$$

Primjetimo da je data jednakost definisana za $\forall\varphi\in(0,2\pi)$

$$D \xrightarrow{\text{transformišemo}} D': \begin{cases} 0\le\varphi\le 2\pi \\ 0\le\rho\le\sqrt{a^2(\cos^4\varphi+\sin^4\varphi)} \end{cases}$$

$$P=\iint_{D'}\rho\,d\rho\,d\varphi=\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2(\cos^4\varphi+\sin^4\varphi)}}\rho\,d\rho=\frac{1}{2}\int_0^{2\pi}a^2(\cos^4\varphi+\sin^4\varphi)\,d\varphi$$

$$\int_0^{2\pi}\cos^4\varphi\,d\varphi=\ldots=\frac{3\pi}{4} \qquad \int_0^{2\pi}\sin^4\varphi\,d\varphi=\ldots=\frac{3\pi}{4}$$

$$P=\frac{a^2}{2}\cdot\frac{\cancel{6}^{\,3}\pi}{\cancel{4}_{\,2}}=\frac{3a^2}{4}\pi$$ tražena površina

(#) Metodom diferenciranja po parametru izračunati integral

$$\int_0^\infty \frac{\operatorname{arctg} \alpha x}{x(1+x^2)}\,dx \quad \text{gdje je } \alpha > 0.$$

Rj. - upute:

$$K(\alpha) = \int_0^\infty \frac{\operatorname{arctg}\alpha x}{x(1+x^2)}\,dx$$

$$\left(\frac{\operatorname{arctg}\alpha x}{x(1+x^2)}\right)'_\alpha = \frac{1}{x(1+x^2)} \cdot \frac{x}{1+\alpha^2 x^2} = \frac{1}{(1+x^2)(1+\alpha^2 x^2)}$$

$$K'(\alpha) = \int_0^\infty \left(\frac{\operatorname{arctg}\alpha x}{x(1+x^2)}\right)'_\alpha dx = \int_0^\infty \frac{dx}{(1+x^2)(1+\alpha^2 x^2)}$$

$$\frac{1}{(1+x^2)(1+\alpha^2 x^2)} = \frac{Ax+B}{1+\alpha^2 x^2} + \frac{Cx+D}{1+x^2} \quad \ldots \Rightarrow \quad A=0 \quad C=0 \quad B = \frac{\alpha^2}{\alpha^2-1} \quad D = \frac{-1}{\alpha^2-1}$$

$$K'(\alpha) = \frac{\alpha^2}{\alpha^2-1}\int_0^\infty \frac{dx}{1+\alpha^2x^2} + \frac{(-1)}{\alpha^2-1}\int_0^\infty \frac{dx}{1+x^2} = \ldots = \frac{\frac{\pi}{2}}{\alpha+1}$$

Dobili smo $K'_\alpha = \frac{\pi}{2} \cdot \frac{1}{\alpha+1} \Rightarrow K = \frac{\pi}{2}\ln(\alpha+1) + C$

Kako je $K(0) = 0 = \frac{\pi}{2}\ln 1 + C \Rightarrow C = 0$

pa imamo $\int_0^\infty \frac{\operatorname{arctg}\alpha x}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(\alpha+1)$ traženo rješenje